Probabilistic Safe Adaptive Merging Control for Autonomous Vehicles under motion uncertainty

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Abstract— **In this work we address the safe adaptive control problem for autonomous vehicles in the highway on-ramp merging scenario. We argue that by developing a Control Barrier Function (CBF)-based method, autonomous vehicles are able to perform adaptive interactions with human drivers with safety guarantees under uncertainty. We propose a novel extension of traditional CBF to a probabilistic setting for stochastic system dynamics with provable chance-constrained safety and provide a theoretical analysis to discuss the solution feasibility guarantee and design factors reflecting different vehicle behaviors. This allows for adaptation to different driving strategies with a formally provable feasibility guarantee for the ego vehicle's safe controller. The results demonstrate the enhanced safety and adaptability of our proposed approach.**

I. Introduction

Since self-driving cars can not replace all the human drivers immediately, they will have to share roads with human drivers for a long time. For autonomous vehicles (AV), how to incorporate uncertainty from a complex scenario and achieve safety and efficiency at the same time has been a very popular research topic. One of the most typical scenarios is the highway on-ramp merging scenario where intensive interactions exist between human drivers and AVs. Collisions will happen if they can not be planned and controlled safely.

Automated Cruise Control (ACC)-like distance control methods has been applied broadly in the car-making industry, by forcing the vehicle to brake when distance is less than the specified minimum safety distance, while maintaining the minimum deviation from the user-defined driving speed. However, vehicle control with ACC in the real world can be difficult due to potential conflicts of multiple objectives. The NHTSA categorizes these methods as convenience features rather than safety features [1].

The goal of this work is to enable the AV to merge with human-driven cars safely and efficiently in the highway on-ramp merging scenario. Specifically, we present an improved Control Barrier Function (CBF)-based approach to allow AVs to perform safe merging under uncertainty with formally provable safety guarantee. Moreover, by leveraging the bi-level optimization structure, we provide a solution feasibility guarantee in runtime which sets us apart from the traditional CBF-based methods. Different design factors for adaptive behavior generation are also discussed.

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II. Related Work

In the area of behavior planning and control of autonomous driving, learning-based methods are widely used. Dong et al. [2] proposed a Probabilistic Graphical Model-based method to help the ego vehicle decide whether to yield or not in ramp merging, followed by an ACC controller. Nishitani et al. [3] introduced a vehicle controller using deep reinforcement learning to improve the merging efficiency while tracking the expected vehicle speed. However, these learning-based methods cannot provide a provably correct safety guarantee which is in critical need.

The CBF-based method was initially proposed by Wieland and Allower [4] to describe an admissible control space that renders forward invariance of a safe set. Notomista et al. [5] proposed a CBF-based method specifically for the two-car racing scenario. Due to the special property of the scenario, driving conservativeness is minimized in order to attain the strongest racing performance, which makes the method not applicable to everyday traffic scenarios.

Quantitative analysis on solution feasibility conditions is another missing gap in existing CBF works. [6] mentions that the solution feasibility can be guaranteed by assuming immediate stop for all robots to prevent collision in the worst case. However, a more principled scheme with explicit theoretical grounding is desirable to automatically decide whether the vehicle needs to apply full braking before it is too late.

To address model uncertainty, [5] and [7] proposed to employ the CBF approach with noisy system dynamics. However, their work assumed that the uncertainty is bounded, which limits the probability distribution and is not suitable for a general consistent solution feasibility guarantee.

III. Background of Traditional CBF

A Control Barrier Functions (CBF) [8] is used to define an admissible control space for safety assurance of dynamical systems. One of its important properties is its forward-invariance guarantee of a desired safety set. Consider the nonlinear system in control affine form: $\dot{x} = f(x) + g(x)u$, where $x \in \mathcal{X} \subset \mathbb{R}^n$ and $u \in \mathcal{U} \subset \mathbb{R}^m$ are the system state and control input with f and g assumed to be locally Lipschitz continuous. A desired safety set $x \in \mathcal{H}$ can be denoted by the following safety function:

$$\mathcal{H} = \{x \in \mathbb{R}^n : h(x) \geq 0\} \tag{1}$$

Thus the control barrier function for the system to remain in the safety set can be defined as follows [8]:

Definition 1. *(Control Barrier Function) Given a dynamical system defined above and the set $\mathcal{H}$ defined in (1) with a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, then h is a control barrier function (CBF) if there exists an extended class $\mathcal{K}_\infty$ function for all $x \in \mathcal{X}$ such that*

$$\sup_{u\in\mathcal{U}} \{L_f h(x) + L_g h(x)u\} \geq -\kappa\big(h(x)\big) \tag{2}$$

where $\dot{h}(x, u) = L_f h(x) + L_g h(x)u$ with $L_f h, L_g h$ as the Lie derivatives of h along the vector fields f and g. Similar to [6], in this paper we use the particular choice of extended class $\mathcal{K}_\infty$ function with the form as $\kappa(h(x)) = \alpha h(x)$ where $\alpha \geq 0$ is a CBF design parameter controlling system behaviors near the boundary of $h(x) = 0$. Hence, the admissible control space in Eq. 2 can be redefined as

$$\mathcal{B}(x) = \{u \in \mathcal{U} : \dot{h}(x, u) + \alpha h(x) \geq 0\ \} \tag{3}$$

It is proved in [8] that any controller $u \in \mathcal{B}(x)$ will render the safe state set $\mathcal{H}$ forward invariant. In this paper, we consider the particular choice of pairwise vehicle safety function $h^s_{em}(x)$, safety set $\mathcal{H}^s$, and admissible safe control space $\mathcal{B}^s(x)$ as follows.

$$\begin{aligned} \mathcal{H}^s &= \{x \in \mathcal{X} :\ h^s_{em}(x) = ||x_e - x_m||^2 - R^2_{safe} \geq 0,\ \forall m\} \\ \mathcal{B}^s(x) &= \{u \in \mathcal{U} :\ \dot{h}^s_{em}(x, u) + \alpha h^s_{em}(x) \geq 0,\ \forall m\} \end{aligned} \tag{4}$$

where x_e, x_m are the states of ego vehicle e and each merging vehicle m with $R_{safe} \in \mathbb{R}$ as the minimum allowed safety distance between pairwise vehicles.

IV. Method

A. Problem Formulation

In this section, the problem formulation in the ramp merging scenario is introduced. The goal is to control the ego vehicle (host vehicle) on the main road to merge safely with the human-driven vehicles (merge cars) on the ramp with motion uncertainty. The system dynamics of a vehicle can be described by double integrators as follows since acceleration plays a key role in the safety considerations.

$$\dot{X} = \begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix} = \begin{bmatrix}0_{2\times2} & I_{2\times2}\\ 0_{2\times2} & 0_{2\times2}\end{bmatrix}\begin{bmatrix}x\\ v\end{bmatrix} + \begin{bmatrix}0_{2\times2} & I_{2\times2}\\ I_{2\times2} & 0_{2\times2}\end{bmatrix}\begin{bmatrix}u\\ \epsilon\end{bmatrix} \tag{5}$$

where $x \in \mathcal{X} \subset \mathbb{R}^2, v \in \mathbb{R}^2$ are the position and linear velocity of each car respectively and $u \in \mathbb{R}^2$ represents the acceleration control input. $\epsilon \sim \mathcal{N}(\hat{\epsilon}, \Sigma)$ is a random Gaussian variable with known mean $\hat{\epsilon} \in \mathbb{R}^2$ and variance $\Sigma \in \mathbb{R}^{2\times2}$, representing the uncertainty in each vehicle's motion. We assume the human-driven merging vehicle's velocity v_m and the motion uncertainty distribution of ϵ_m are known to the ego vehicle per time step with $u_m = 0$.

While performing safe merging with human vehicles, the ego vehicle is expected to maintain task efficiency, passing the merging point as fast as possible. Therefore, the objective function can be formulated as a quadratic programming problem for the ego vehicle with the control input u_e.

$$\begin{aligned} &\min_{u_e\in\mathcal{U}_e} ||u_e - \bar{u}||^2 \\ s.t\quad & U_{min} \leq u_e \leq U_{max} \\ & \Pr\Big(\dot{h}^s_{em}(x, u) + \alpha h^s_{em}(x) \geq 0\Big) \geq \eta, \qquad \forall m \end{aligned} \tag{6}$$

where $\bar{u}$ is the nominal expected acceleration for the ego vehicle to follow, and U_{max} and U_{min} are the ego vehicle's maximum and minimum allowed acceleration. R_s is the minimum allowed distance between two vehicles to avoid collision for safety. We consider the chance-constrained optimization problem to accommodate uncertainty with $\eta \in (0, 1)$ as the desired confidence of probabilistic safety. $\Pr(\cdot)$ denotes the probability of a condition to be true. We employ the chance constraints over vehicle controller u_e to ensure the resulting lower-bounded probability of vehicles being collision-free. Most of prior CBF works assume a fixed parameter α, however, a fixed α could make Eq. 6 not solvable under certain circumstances [6]. One of the main contributions of our work is formulating the original problem Eq. 6 as the following bi-level optimization process with two layers as Eq. 7: one for optimization over u_e, and the other one for optimization over α for feasibility guarantee.

$$\begin{aligned} &\min_{u_e\in\mathcal{U}_e, \alpha\in\mathcal{A}} ||u_e - \bar{u}||^2 \\ s.t\quad & U_{min} \leq u_e \leq U_{max} \\ & \Pr\Big(\dot{h}^s_{em}(x, u) + \alpha h^s_{em}(x) \geq 0\Big) \geq \eta, \qquad \forall m \\ & \alpha = \arg\min_{\alpha\in\mathcal{A}}\{||\alpha - \bar{\alpha}||^2\} \end{aligned} \tag{7}$$

where $\mathcal{A}$ is the feasible set for α that will be proved to ensure solution feasibility of u_e. $\bar{\alpha}$ is a nominal value by user to specify the desired conservativeness of the safe behavior. The feasible set $\mathcal{A}$ changes over time and to ensure solution feasibility of u_e, the goal is to ensure the set $\mathcal{A}$ is consistently non-empty so that we can always find an α that causes a nonempty set of u_e (if it exists) to satisfy the safety constraint.

B. Active and Feasible Condition of CBF

This section will first present a novel approach for CBF with probabilistic safety consideration under uncertainty and discuss the feasibility analysis with the CBF constraints.

Theorem 2. *Given a stochastic dynamical system in Eq. 5 and a confidence level $\eta \in (0, 1)$, the following admissible control space $\mathcal{B}^s_\eta(x)$ ensures a chance-constrained safety condition in Eq. 7 for the ego vehicle with each merging car m.*

$$\begin{aligned} &\mathcal{B}^s_\eta(x) = \{u_e \in \mathcal{U}_e : A_{em}u_e \geq b_{em}, \quad \forall m\} \\ &A_{em} = -2\Delta x^T_{em}\Delta t, \quad b_{em} = 2\Delta x^T_{em}(\Delta v_{em} + \Delta\hat{\epsilon}_{em}) + \alpha h^s_{em}(x) \\ &\qquad\qquad - \Phi^{-1}(\eta)\sqrt{\Delta x^T_{em}\Delta\Sigma_{em}\Delta x_{em}} \end{aligned} \tag{8}$$

where $\Delta x_{em} = x_e - x_m, \Delta v_{em} = v_e - v_m, \Delta\epsilon_{em} = \epsilon_e - \epsilon_m \sim \mathcal{N}(\Delta\hat{\epsilon}_{em}, \Delta\Sigma_{em})$ for ego vehicle e and each

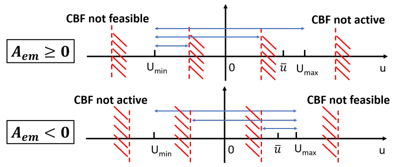

Fig. 1: Illustration of CBF solution feasibility. The red dashed lines stand for the value of K_1 and K_2 in two cases. Blue intervals represent the reducing solution set while α keep decreasing. **Case 1:** $A_{em} \geq 0$. We have $u_e \leq K_1 = A_{em}^{-1} b_{em}$, which provides an upper bound for u_e. While α keeps decreasing, CBF is activated when K_1 overlaps with U_{max}, and infeasibility is about to happen when K_1 overlaps with U_{min}. **Case 2:** $A_{em} < 0$. We have $u_e \geq K_2 = A_{em}^{-1} b_{em}$. While α keeps decreasing, we observe that CBF is activated at the overlapping point with U_{min}, and infeasibility is about to happen at the overlapping point with U_{max}.

merging vehicle m. $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function (CDF) of the standard zero-mean Gaussian distribution with unit variance. Proof can be found in [9].

Next, we will discuss the solution feasibility conditions of the formulated problem in Eq. 7. In particular, we will present how to ensure non-emptiness of α set $\mathcal{A}$ for non-emptiness of set $u_e \in \mathcal{B}_\eta^s(x)$ that preserves the forward invariant safety. Given Eq. 8, the set $\mathcal{A}$ feasibility analysis is decomposed into two situations based on the positiveness of A_{em}. The active and feasible conditions of CBF depend on the overlap set between the CBF solution set and the bounded control constraints as shown in Fig. 1. To simplify the discussion, here we assume $u_e = R_\theta a_e \in \mathbb{R}^2$ is determined by ego vehicle's linear acceleration $a_e \in \mathbb{R}$ along the ramp and the rotation matrix $R_\theta \in SO(2)$ by the road geometry. Thus we reformulate Eq. 8 by $A_{em}(R_\theta a_e) \leq b_{em}$ and redefine $A_{em} = A_{em} R_\theta \in \mathbb{R}, u_e = a_e \in \mathbb{R}$.

In conclusion, the boundary conditions of the CBF are:

$$\begin{aligned} A_{em} \geq 0: \quad & \alpha^m_{feasible} = M_m U_{min} + N_m, \alpha_{active} = M_m U_{max} + N_m \\ A_{em} < 0: \quad & \alpha^m_{feasible} = M_m U_{max} + N_m, \alpha_{active} = M_m U_{min} + N_m \end{aligned} \tag{9}$$

$$\begin{aligned} M_m = \frac{A_{em}}{h^s_{em}}, N_m = \frac{T_m}{h^s_{em}}, T_m = -2\Delta x_{em}^T(\Delta v_{em} + \Delta\hat{\epsilon}_{em}) \\ + \Phi^{-1}(\eta)\sqrt{\Delta x_{em}^T \Delta\Sigma_{em} \Delta x_{em}} \end{aligned} \tag{10}$$

C. Consistent Solution Feasibility Guarantee

In the previous section, the relationship between α and the solution feasibility was analyzed, and explicit feasible conditions on α were given. Here, a Safe Adaptive Algorithm (Algorithm 1) is introduced for guaranteed solution feasibility. For time steps 1 to N, at each time step t, u_e^t is calculated through the first-layer optimization. Then given states of both vehicles, α^{t+1}_{fea} is calculated to ensure the feasible solution set $\mathcal{B}_\eta^s(x)$ is non-empty at $t+1$. The second-layer optimization is performed and α is updated at each iteration. The advantage of this algorithm, compared to fixing the α value, is that it provides a dynamic solution feasibility guarantee at run time.

Remark 1. *The problem can still be infeasible with our proposed method if the initial conditions make it impossible to ensure safety, e.g. the ego vehicle is driving too fast, and it's already too late to avoid collision, and no matter what α we choose, Eq. 3 can never be satisfied. However, the proposed method does guarantee solution feasibility as long as such a solution exists.*

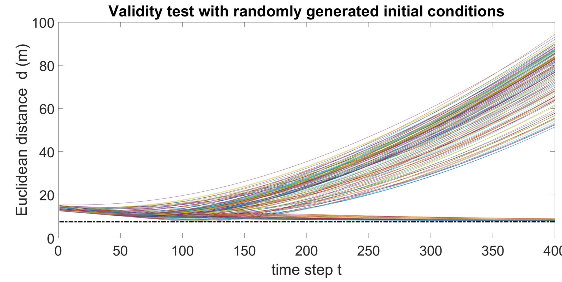

Fig. 2: Validity test of the proposed method. The black dashed line stands for the minimum allowed safety distance $R_{safe} = 8$m. The confidence level η is set to be 99%. The two different kinds of curve shapes correspond to two merging results: asymptotically approaching R_{safe} indicates merging after the merging vehicle and increasing Euclidean distance indicates merging in front of the merging vehicle.

Algorithm 1 Safe Adaptive Merging Algorithm

input: $\Delta x_{em}, \Delta v_{em}, \Delta t, R_{safe}, a_m, \bar{\alpha}$
output: α, u_e
 for $t = 1 : N$ **do**
  compute A^t_{em}, b^t_{em}
  $u^t_e = \arg\min_{u_e} ||u^t_e - \bar{u}||^2$
  compute $A^{t+1}_{em}, M^{t+1}_{em}, N^{t+1}_{em}$ via forward kinematics
  if $A^{t+1} \geq 0$ **then**
   $\alpha^{t+1}_{fea} = M^{t+1}_{em} U_{min} + N^{t+1}_{em}$
  else
   $\alpha^{t+1}_{fea} = M^{t+1}_{em} U_{max} + N^{t+1}_{em}$
  end if
  $\alpha^{t+1} = \arg\min_\alpha ||\alpha - \max(\bar{\alpha}, \alpha^{t+1}_{fea})||^2, \forall m$
 end for

V. Experiment & Discussion

Validity Test: To prove that the proposed method is valid, we conduct experiments against one merging vehicle, with randomly generated ego vehicle initial conditions, including position and velocity and desired driving strategy α. The results are shown in Fig. 2. It is observed that all 400 trials keep the minimum distance as required, and the collision rate is 0%. To better illustrate the advantage of the proposed method, a comparison with traditional CBF with fixed α is made as shown in Fig. 3. The proposed method updates α in the green zone, while traditional CBF does not, which leads to solution infeasibility from $t = 77$ to $t = 194$ and violation of the minimum safety distance requirement. The proposed method maintains solution feasibility consistently and performs the merging safely.

Vehicle behavior factors-Effect of the CBF parameter α: The choice of the CBF parameter α is a key factor in shaping a vehicle's behavior. Generally, the larger α is, the more admissive action space the ego vehicle will have. To verify this statement, ego vehicle behaviors with different α values are compared. For better visualization effect, we observe ego vehicle merging control with a single merging

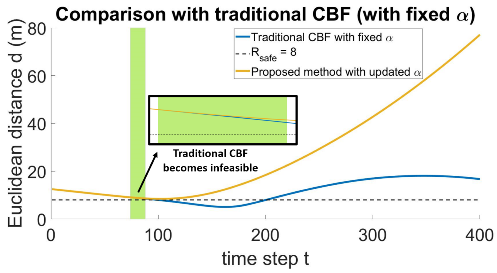

Fig. 3: Comparison of the proposed method with traditional CBF with fixed α. The green zone indicates the time interval when α is updated in the proposed method to guarantee solution feasibility and therefore safety. The traditional CBF solution with fixed α becomes infeasible starting from this interval and eventually leads to collision with distance smaller than $R_{safe} = 8$.

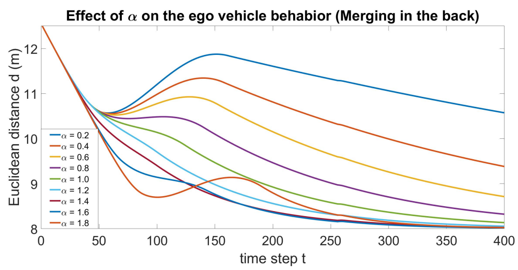

Fig. 4: How α affects the ego vehicle's behavior (Merging behind). The smaller α is, the earlier the ego vehicle will brake to keep the distance strictly. Larger α will allow the ego vehicle to approach the merging vehicle more quickly, and to brake as late as possible.

vehicle, while the initial conditions of the ego vehicle and the profile of the merging vehicle are kept the same to make the comparison fair. The results are shown in Fig. 4 and Fig. 5 (R_{safe} = 8 m). Before $t = 40$ (Fig. 4) and $t = 100$ (Fig. 5), all trials share exactly the same states, meaning the CBF is not active yet, and therefore the value of α does not make any difference in the ego vehicle's behavior. After those times, the ego vehicle decides to decelerate or accelerate at a certain point. The smaller α is, the earlier the deceleration or acceleration decision is made to avoid collision in future steps. In other words, the larger α is, the more aggressive the driving strategy is for the ego vehicle, and the deceleration as a precautionary action is more and more delayed.

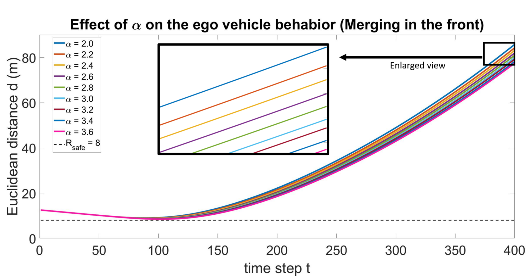

Fig. 5: How α affects the ego vehicle's behavior (Merging in the front). While the ego vehicle passes the merging vehicle around $t = 90$ in Fig. 5, smaller α makes the ego vehicle accelerate as early as possible to prevent getting too close to the merging vehicle in the future, and larger α tends to make the ego drive with $\bar{u}$ as long as possible while getting closer to the merging vehicle, and only increases acceleration when necessary.

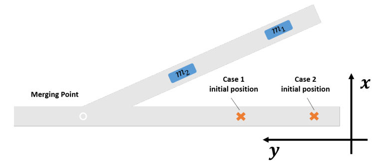

Fig. 6: Illustration of the ego vehicle's initial positions in two cases, where m_1, m_2 represent the first and the second merging vehicle. The behavior of the two merging vehicles remains the same in the two cases. where m_1 and m_2 have constant velocity. As the initial condition, $v_{m_2} > v_{m_1}$. The minimum allowed safety distance R_{safe} is set to 5 m.

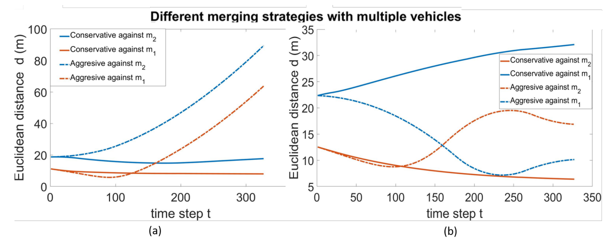

Fig. 7: Comparison of different merging strategies for case 1 (a) and case 2 (b). In case 1, $v_e = v_{m1} + 2$. While performing conservative merging, with α = 1, the ego vehicle merges in between m_1 and m_2 safely. The ego vehicle keeps the distance to both of the merging vehicles without much change. While performing aggressive merging, with $\alpha = 15$, the ego vehicle merges in front of m_2 safely. The ego vehicle accelerates obviously and completes the merging around $t = 95$, and the distance to both merging vehicles increases after that. In case 2, $v_e = v_{m_1} - 2$. While performing conservative merging, with α = 1, the ego vehicle merges after m_1 safely. The ego vehicle accelerates while maintaining the required safety distance. While performing aggressive merging, with $\alpha = 15$, the ego vehicle merges in between the two merging vehicles safely. The ego vehicle accelerates to pass m_1 and merges in between m_1 and m_2. It keeps approaching m_2 until it decreases its acceleration to meet the future safety guarantee.

Vehicle behavior factors-Effect of initial conditions: The initial conditions also affect the merging behavior of the ego vehicle. Consider the merging control with two merging vehicles, shown in Fig. 6 with three different slots available to merge in. Whether the ego vehicle can merge into any of the three slots depends on the initial conditions, including relative distance and speed, as well as the nominal acceleration $\bar{u}$ set for the ego vehicle to follow and the minimum allowed safety distance R_{safe}. Intuitively, if R_{safe} is set to be very large, the ego vehicle has to keep far enough from both of the merging vehicles, and that makes it difficult for it to squeeze into the gap between the two merging vehicles without breaking the minimum safety distance requirement. Besides, the relative distance and relative speed also matter. Together with $\bar{u}$, they decide the reachability set for the ego vehicle, which is the set of positions the ego vehicle can achieve. We take a look at two specific cases for detailed illustration. The experimental results are shown in Fig. 7. The ego vehicle's initial positions for the two cases are shown in Fig. 6.

VI. Conclusion

We present an adaptive merging control algorithm for AVs in the ramp merging scenario with probabilistic safety guarantee. Experiments with different conditions are used to demonstrate the power of CBF in applying different driving strategies to the ego vehicle via a single parameter α. The proposed method provides a theoretically consistent solution feasibility analysis with explicit bounds on the CBF parameter α. In future work, we plan to combine the proposed framework with learning-based methods that use real-world datasets to realize safe control with a data-driven approach to determine the appropriate strategy.

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